Additional homework problems

1. Let F be any field. Prove that for every x in F, $x \cdot 0 = 0$. Do this using just the properties listed on pages 2 and 3 of the textbook.

   Proof: We start from $0 + 0 = 0$, which is true because 0 is the additive identity. Multipliying both sides by x, we conclude that $x(0 + 0) = x0$. The distributive property allows us to expand the left side, so we find that $x0 + x0 = x0$. At this point all we know is that $x0$ is some element of F, but in any case it has an additive inverse, which we could call $-(x0)$. Then adding this to both sides of the equality, we find that $(x0 + x0) + -(x0) = x0 + -(x0)$. The right side is 0, by the definition of the additive inverse. Using the associative law, we can rewrite the left side, obtaining $x0 + (x0 + -(x0)) = 0$. But $(x0 + -(x0) = 0$, so the equation becomes $x0 + 0 = 0$. Since 0 is the additive identity, we conclude that $x0 = 0$.

2. Prove that if x is any element of a field F, then $(-x)(-x) = x^2$. Again, just use the properties listed on pages 2 and 3 of the textbook.

   I will write this somewhat more succintly. We start with the fact that $x + -x = 0$, and use the previous result, the distributive, associative, and commuative laws:

$$\begin{aligned}
x + -x &= 0 \\
(x + -x)(-x) &= 0(-x) \\
x(-x) + (-x)(-x) &= 0 \\
xx + x(-x) + (-x)(-x) &= xx + 0 \\
x(x + -x) + (-x)(-x) &= xx \\
x0 + (-x)(-x) &= xx \\
0 + (-x)(-x) &= xx \\
(-x)(-x) &= xx
\end{aligned}$$

3. Let F_2 be the field with two elements. Find a polynomial of degree 2 with coefficients in F_2 that has no roots in F_2. (Hint: Use trial and error if necessary.)

   Solution: $p := x^2 + x + 1$ has no roots in F_2. Indeed, $p(0) = 1$ and $p(1) = 1 + 1 + 1 = 1$, and there are no other possibilities.